Series expansion in fractional calculus and fractional differential equations

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Abstract

Fractional calculus is the calculus of differentiation and integration of non-integer orders. In a recent paper (Annals of Physics **323** (2008) 2756-2778), the Fundamental Theorem of Fractional Calculus is highlighted. Based on this theorem, in this paper we introduce fractional series expansion method to fractional calculus. We define a kind of fractional Taylor series of infinitely fractionally-differentiable functions. By using this kind of fractional Taylor series, we give a fractional generalization of hypergeometric functions and derive corresponding differential equations. For finitely fractionally-differentiable functions, we observe that the non-infinitely fractionally-differentiability is due to more than one fractional indices. We expand functions with two fractional indices and illustrate how this kind of series expansion can help to solve fractional differential equations.

Keywords:

Fractional calculus;

Fundamental Theorem of Fractional Calculus;

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Fractional hypergeometric function;

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1 Introduction

Fractional calculus is the calculus of differentiation and integration of non-integer orders. During last three decades or so, fractional calculus has gained much attention due to its demonstrated applications in various fields of science and engineering [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. There are many good textbooks of fractional calculus and fractional differential equations, such as [1, 2, 3, 4, 5]. For various applications of fractional calculus in physics, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and references therein.

In Calculus, the Fundamental Theorem of Calculus (Newton-Leibniz Theorem) is of fundamental importance, matching its name. For fractional calculus, an analogous theorem has been highlighted recently in a paper [18]. When trying to construct a consistent fractional vector calculus, V. E. Tarasov observed that many of fundamental problems can be solved by using a generalization of the Fundamental Theorem of Calculus.

Series expansion is an important tool to calculus. Particularly, series expansion plays an important role in solving some differential equations, such as the hypergeometric differential equations [19, 20, 21]. However, fractional series expansion has not yet been introduced to fractional calculus.

In this paper, based on the Fundamental Theorem of Fractional Calculus, we will introduce fractional series expansion method to fractional calculus. We will define a kind of fractional Taylor series of infinitely fractionally-differentiable functions. By using of this kind of fractional Taylor series, we will give a fractional generalization of hypergeometric functions and derive corresponding differential equations. For finitely fractionally-differentiable functions, we observe that the non-infinitely fractionally-differentiability is due to more than one fractional indices. We will expand functions with two fractional indices and illustrate how this kind of series expansion can help to solve fractional differential equations.

The structure of this paper is as follows. In Section 2, we briefly review fractional derivative, fractional integral and the Fundamental Theorem of Fractional Calculus. In Section 3, we introduce the fractional Taylor series of an infinitely fractionally-differentiable function and give some examples. In Section 4, we make a generalization of the hyper-

geometric functions. In Section 5, we discuss finitely fractionally-differentiable functions. In Section 6, we give our summary.

2 Fractional Caculus

In this section, we briefly review the definitions of fractional integral and fractional derivative, and the Fundamental Theorem of Fractional Calculus. For more details, see [1, 2, 3, 4, 5].

2.1 Fractional integral and fractional derivative

There are many ways to define fractional derivative and fractional integral. Most of them are based on the idea that we can generalize the equation

$$\frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \qquad n \in N, \tag{1}$$

by replacing each factorial by a gamma function, to

$$\frac{d^\alpha x^\beta}{dx^\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad \alpha > 0. \tag{2}$$

In terms of integral operation I, the idea is to generalize

$$I^n x^m = \frac{m!}{(m+n)!} x^{m+n} \tag{3}$$

to

$$I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}. \tag{4}$$

The most commonly-used fractional integral is the Riemann-Liouville fractional integral (RLFI).

Let $f(x)$ be a function defined on the interval $[a,b]$. Let α be a positive real.

The right Riemann-Liouville fractional integral (right-RLFI) is defined by

$$I^\alpha_{a|x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \tag{5}$$

and the left Riemann-Liouville fractional integral (left-RLFI) is defined by

$$I^\alpha_{x|b} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi) d\xi. \tag{6}$$

We note that in this paper our use of notations "right" and "left" is different from the common use, for reasons that will be clear later.

For fractional derivatives, the Caputo fractional derivative (CFD) is a commonly-used one. Let $n \equiv [\alpha] + 1$.

The right-CFD and left-CFD are defined, respectively by

$$D_{a|x}^{C,\alpha} f(x) = I_{a|x}^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \tag{7}$$

$$D_{x|b}^{C,\alpha} f(x) = I_{x|b}^{n-\alpha} (-\frac{d}{dx})^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (\xi - x)^{n-\alpha-1} (-\frac{d}{d\xi})^n f(\xi) d\xi. \tag{8}$$

One can check that the above definitions really generalize (1) and (3), and give

$$D_{a|x}^{C,\alpha} (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, \quad \beta \neq 0, 1, ..., [\alpha]; \tag{9}$$

$$D_{x|b}^{C,\alpha} (b-x)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (b-x)^{\beta-\alpha}, \quad \beta \neq 0, 1, ..., [\alpha]; \tag{10}$$

$$I_{a|x}^{\alpha} (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\beta+\alpha}; \tag{11}$$

$$I_{x|b}^{\alpha} (b-x)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (b-x)^{\beta+\alpha}. \tag{12}$$

Especially, the Caputo fractional derivative on a constant ($\beta = 0$) yields zero,

$$D_{a|x}^{C,\alpha} \cdot 1 = 0, \tag{13}$$

$$D_{x|b}^{C,\alpha} \cdot 1 = 0. \tag{14}$$

This simple property is decisive in the fractional series expansion and in our preference of the Caputo fractional derivative to another commonly-used fractional derivative, the Riemann-Liouville fractional derivative, whose operation on a constant gives not zero,

$$D_{a|x}^{RL,\alpha} \cdot 1 = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1, \tag{15}$$

$$D_{x|b}^{RL,\alpha} \cdot 1 = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1. \tag{16}$$

2.2 Fundamental Theorem of Fractional Calculus

The Fundamental Theorem of Calculus (Newton-Leibniz Theorem) is

$$\int_a^b dx \frac{d}{dx} f(x) = f(b) - f(a), \tag{17}$$

$$\frac{d}{dx} \int_a^x f(\xi) d\xi = f(x). \tag{18}$$

This theorem means that the derivative operation is inverse to the integral operation, and vice versa.

In fractional calculus, an analogous theorem exists [5, 18].

Fundamental Theorem of Fractional Calculus.

(1) Let $\alpha > 0$ and let $f(x) \in L_\infty(a,b)$ or $f(x) \in C[a,b]$. Then

$$D^{C,\alpha}_{a|x} I^{\alpha}_{a|x} f(x) = f(x), \tag{19}$$

$$D^{C,\alpha}_{x|b} I^{\alpha}_{x|b} f(x) = f(x). \tag{20}$$

(2) Let $0 < \alpha < 1$. If $f(x) \in AC[a,b]$ or $f(x) \in C[a,b]$. Then

$$I^{\alpha}_{a|x} D^{C,\alpha}_{a|x} f(x) = f(x) - f(a), \tag{21}$$

$$I^{\alpha}_{x|b} D^{C,\alpha}_{x|b} f(x) = f(x) - f(b). \tag{22}$$

Here $L_\infty(a,b)$ is the set of those Lebesgue measurable functions f on (a,b) for which $\|f\|_\infty < \infty$, where $\|f\|_\infty = \text{ess sup}_{a \le x \le b}|f(x)|$. Here ess sup$|f(x)|$ is the essential maximum of the function $|f(x)|$. $AC[a,b]$ is the space of functions f which are absolutely continuous on $[a,b]$. $C[a,b]$ is the space of continuous functions f on $[a,b]$ with the norm $\|f\|_C = \max_{x \in [a,b]} |f(x)|$.

The proof of this theorem can be obtained from [5], in which these results are included in Lemma 2.21 and Lemma 2.22 there.

So, by this theorem, one can say that the right (left) Caputo fractional derivative operation and the right (left) Riemann-Liouville fractional integral operation are inverse to each other.

3 Fractional Taylor series of infinitely fractionally-differentiable functions

In this section, we will introduce fractional series expansion method to fractional calculus and define a kind of fractional Taylor series.

We observe that

$$f(x) = f(a) + D^{\alpha}_{a|x} f(x)\big|_{x=\xi} \cdot (I^{\alpha}_{a|x} \cdot 1), \tag{23}$$

where $a < \xi < x$, and ξ varies with the integral upper bound. Here and after, $D^{\alpha}_{a|x}$ denotes the right Caputo fractional derivative $D^{C,\alpha}_{a|x}$.

The corresponding formula in integer-order calculus is

$$f(x) = f(a) + \frac{df}{dx}\bigg|_{x=\xi} \cdot (x-a), \tag{24}$$

which is the Lagrange Mean Value Theorem.

Make a step further,

$$f(x) = f(a) + D^{\alpha}_{a|x} f(x)\big|_{x=a} \cdot (I^{\alpha}_{a|x} \cdot 1) + D^{\alpha}_{a|x} D^{\alpha}_{a|x} f(x)\big|_{x=\xi} \cdot (I^{\alpha}_{a|x} I^{\alpha}_{a|x} \cdot 1). \tag{25}$$

The corresponding formula in integer-order calculus is

$$f(x) = f(a) + \frac{df}{dx}\bigg|_{x=a} \cdot (x-a) + (\frac{d}{dx})^2 f\bigg|_{x=\xi} \cdot \frac{1}{2}(x-a)^2. \tag{26}$$

And so on. One can extend this expansion to infinite order if the function is sufficiently smooth.

Based on this observation a definition of a formal fractional Taylor series expansion can be made.

Definition 1.a. Let $f(x)$ be a function defined on the right neighborhood of a, and be an infinitely fractionally-differentiable function at a, that is to say, all $(D^{\alpha}_{a|x})^m f(x) (m = 0, 1, 2, 3..)$ exist, and are not singular at a. The formal fractional right-RL Taylor series of a function is

$$f(x) = \sum_{m=0}^{\infty} (D^{\alpha}_{a|x})^m f(x)\big|_{x=a} \cdot [(I^{\alpha}_{a|x})^m \cdot 1]. \tag{27}$$

Explicitly,

$$(I^{\alpha}_{a|x})^m \cdot 1 = \frac{1}{\Gamma(m\alpha+1)}(x-a)^{m\alpha}. \tag{28}$$

Definition 1.b. Let $f(x)$ be a function defined on the left neighborhood of b, and be an infinitely fractionally-differentiable function at b, that is to say, all $(D^{\alpha}_{x|b})^m f(x)(m = 0, 1, 2, 3..)$ exist, and are not singular at b. The formal fractional left-RL Taylor series of a function is

$$f(x) = \sum_{m=0}^{\infty} (D^{\alpha}_{x|b})^m f(x)\big|_{x=b} \cdot [(I^{\alpha}_{x|b})^m \cdot 1]. \tag{29}$$

Explicitly,

$$(I^{\alpha}_{x|b})^m \cdot 1 = \frac{1}{\Gamma(m\alpha + 1)}(b - x)^{m\alpha}. \tag{30}$$

In the above definitions, $D^{\alpha}_{a|x}$ is the right Caputo fractional derivative $D^{C,\alpha}_{a|x}$; $D^{\alpha}_{x|b}$ is the left Caputo fractional derivative $D^{C,\alpha}_{x|b}$. $I^{\alpha}_{a|x}$ and $I^{\alpha}_{x|b}$ are right- and left- Riemann-Liouvelle fractional integral, respectively.

Remark 1. One can easily check the formal correctness of the expansions by using of the Fundamental Theorem of Fractional Calculus, or the relations (9)-(14). For rigorous validity, convergence is required.

Remark 2. Series expansion has played an important role in Calculus, particularly in solving differential equations. However, fractional series expansion has not yet been introduced to fractional calculus. This is because a pre-requisite that makes fractional series expansion possible is the Fundamental Theorem of Fractional Calculus, which is only recently proved and highlighted [5, 18].

Remark 3. We may expect that fractional series expansion will shed new light on fractional calculus, especially the field of fractional differential equations. In the next section, we will use this expansion to define a fractional generalization of hypergeometric functions and discuss their differential equations.

Remark 4. The fractional Taylor series is defined for infinitely fractionally-differentiable functions. Finitely fractionally-differentiable functions will be discussed in Section 5.

In the following, we give some simple examples of fractional Taylor series.

Example 1. $(x - a)^{\beta}$, with no $l \in N$ satisfying $\beta = l\alpha$.

$$(D^{\alpha}_{a|x})^m (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - m\alpha + 1)}(x - a)^{\beta - m\alpha}, \tag{31}$$

For large m, the derivative will be singular at a. So we cannot make the expansion.

Example 2.

$$e^{(x-a)^\alpha} = \sum_{m=0}^{\infty} \frac{1}{m!}(x-a)^{m\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(m\alpha+1)}{m!} \frac{1}{\Gamma(m\alpha+1)}(x-a)^{m\alpha}. \tag{32}$$

Example 3. The Mittag-Leffler function $E_\alpha((x-a)^\alpha)$, which satisfies

$$D^\alpha_{a|x} E_\alpha((x-a)^\alpha) = E_\alpha((x-a)^\alpha), \qquad E_\alpha(0) = 1, \tag{33}$$

It is the fractional analogue of $exp(x)$. For arbitrary m, $(D^\alpha_{a|x})^m E_\alpha((x-a)^\alpha)|_{x=a} = 1$. So,

$$E_\alpha((x-a)^\alpha) = \sum_{m=0}^{\infty} [(I^\alpha_{a|x})^m \cdot 1] = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\alpha+1)}(x-a)^{m\alpha}. \tag{34}$$

Notice the difference between $e^{(x-a)^\alpha}$ and $E_\alpha((x-a)^\alpha)$.

Example 4. $cos_\alpha((x-a)^\alpha)$ and $sin_\alpha((x-a)^\alpha)$.

$$cos_\alpha((x-a)^\alpha) = 1 - \frac{(x-a)^{\alpha\cdot 2}}{\Gamma(\alpha\cdot 2+1)} + \frac{(x-a)^{\alpha\cdot 4}}{\Gamma(\alpha\cdot 4+1)} - \frac{(x-a)^{\alpha\cdot 6}}{\Gamma(\alpha\cdot 6+1)} + ... \tag{35}$$

$$sin_\alpha((x-a)^\alpha) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(x-a)^{\alpha\cdot 3}}{\Gamma(\alpha\cdot 3+1)} + \frac{(x-a)^{\alpha\cdot 5}}{\Gamma(\alpha\cdot 5+1)} - ... \tag{36}$$

They satisfy

$$D^\alpha_{a|x} sin_\alpha((x-a)^\alpha) = cos_\alpha((x-a)^\alpha), \tag{37}$$

$$D^\alpha_{a|x} cos_\alpha((x-a)^\alpha) = -sin_\alpha((x-a)^\alpha). \tag{38}$$

4 Fractional hypergeometric function

In the section, we will define a fractional generalization of the hypergeometric functions.

Let us first consider a fractional generalization of the confluent hypergeometric differential equation:

$$z^\alpha (D^\alpha_{0|z})^2 y + (c - z^\alpha) D^\alpha_{0|z} y - ay = 0. \tag{39}$$

Here a and c are complex parameters. When $\alpha = 1$, this is the ordinary confluent hypergeometric equation.

Introducing the fractional Taylor series

$$y(z) = \sum_{k=0}^{\infty} c_k z^{\alpha\cdot k}, \tag{40}$$

and substituting, we get the ratio of successive coefficients

$$\begin{aligned} \frac{c_{k+1} \cdot \Gamma(k\alpha+\alpha+1)}{c_k \cdot \Gamma(k\alpha+1)} &= \frac{a+\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}}{c+\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}}, && (41) \\ \frac{c_1 \cdot \Gamma(\alpha+1)}{c_0} &= \frac{a}{c}. && (42) \end{aligned}$$

Thus we get a solution of the above differential equation,

$$y(z) = \sum_{k=0}^{\infty} \frac{(a)_k^\alpha}{(c)_k^\alpha} \frac{1}{\Gamma(k\alpha+1)} z^{\alpha \cdot k}. \tag{43}$$

Here $(a)_k^\alpha$ is defined as

$$\begin{aligned} (a)_0^\alpha &= 1, \qquad (a)_1^\alpha = a, \\ (a)_k^\alpha &= \left(a+\frac{\Gamma(k\alpha-\alpha+1)}{\Gamma(k\alpha-2\alpha+1)}\right)\left(a+\frac{\Gamma(k\alpha-2\alpha+1)}{\Gamma(k\alpha-3\alpha+1)}\right)...(a)_1^\alpha, \quad k \geq 2. \end{aligned} \tag{44}$$

This can be seen as a fractional generalization of the rising factorial

$$(a)_k = (a+k-1)(a+k-2)...a. \tag{45}$$

And the series (43) can be seen as a generalization the confluent hypergeometric function. If $\alpha = 1$, it is exactly the confluent hypergeometric function.

For the fractional Gauss hypergeometric function, consider the following series

$$y(z) = \sum_{k=0}^{\infty} \frac{(a)_k^\alpha (b)_k^\alpha}{(c)_k^\alpha} \frac{1}{\Gamma(k\alpha+1)} z^{\alpha \cdot k}, \tag{46}$$

which reduces to the Gauss hypergeometric function when $\alpha = 1$.

The ratio of successive coefficients is

$$\frac{c_{k+1} \cdot \Gamma(k\alpha+\alpha+1)}{c_k \cdot \Gamma(k\alpha+1)} = \frac{\left(a+\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)\left(b+\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)}{\left(c+\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\right)}. \tag{47}$$

Making some manipulation, one can get

$$\begin{aligned} & c_{k+1} \cdot c\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)} + c_{k+1} \cdot \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha-\alpha+1)} \\ &= c_k \cdot ab + c_k \cdot (a+b)\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)} + c_k \cdot \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-\alpha+1)}. \end{aligned} \tag{48}$$

This equation can be translated to a fractional differential equation

$$
\begin{aligned}
&ab \cdot f(z) + (a+b)(z-z_0)^\alpha D^\alpha_{z_0|z} f(z) + (z-z_0)^\alpha D^\alpha_{z_0|z}\big[(z-z_0)^\alpha D^\alpha_{z_0|z} f(z)\big] \\
&= c \cdot D^\alpha_{z_0|z} f(z) + (z-z_0)^\alpha (D^\alpha_{z_0|z})^2 f(z).
\end{aligned} \tag{49}
$$

When $\alpha = 1$, this equation reduces to the ordinary Gauss hypergeometric equation. One can check $y(z-z_0)$ defined in (46) satisfies this equation.

For generalized fractional hypergeometric series

$$
y(z) = \sum_{k=0}^{\infty} \frac{(a_1)^\alpha_k \ldots (a_p)^\alpha_k}{(b_1)^\alpha_k \ldots (b_q)^\alpha_k} \frac{1}{\Gamma(k\alpha+1)} z^{\alpha\cdot k}, \tag{50}
$$

making repeated use of $(z-z_0)^\alpha D^\alpha_{z_0|z}$, one can also get its differential equation.

5 Functions with two fractional indices

Not all functions are infinitely fractionally-differentiable, so it is meaningful to investigate finitely fractionally-differentiable functions. In Example 1 in Section 3, we have given an example function that is not infinitely fractionally-differentiable.

We observe that the non-infinitely fractionally-differentiability is due to another fractional index β (no l satisfying $\beta = l\alpha$). A function $f(x)$ is said to have the behavior of fractional index α in the right neighborhood of a, if it can be expanded as a series with the basis $\{(x-a)^{m\alpha} | m = 0, 1, 2, 3 \ldots\}$.

We also observe that some functions can be expanded into a form with two fractional indices, i.e.

$$
f(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} (x-a)^{m\alpha+n\beta}. \tag{51}
$$

These functions can reduce to a function that is infinitely α-differentiable or infinitely β-differentiable, but generally they are finitely fractionally-differentiable functions.

When we are solving fractional differential equations, we should take care if $[\alpha] > 0$ and $\beta \in N$, for the reason of the relation (9). However, for $[\alpha] = 0$ or β not an integer, the above series is fairly a good ansatz.

A function $f(x)$ is said to be $N \cdot \alpha$ fractionally-differentiable at a, if $(D^{\alpha}_{a|x})^m f(x)|_{x=a}$ is finite for $m \leq N$, but infinite for $m = N+1$. Then it could be expanded as

$$\begin{aligned} f(x) &= c_{00} + c_{10}(x-a)^{\alpha} + +c_{20}(x-a)^{2\alpha} + ... + c_{N0}(x-a)^{N\alpha} \\ &+ \sum_{N\alpha < m\alpha + n\beta < (N+1)\alpha} c_{mn}(x-a)^{m\alpha+n\beta} + \sum_{m\alpha+n\beta \geq (N+1)\alpha} c_{mn}(x-a)^{m\alpha+n\beta}, \quad (52) \end{aligned}$$

in which the first term in the second line should not be zero.

Now let us see how this kind of series expansion could help to solve fractional differential equations.

Consider the following equation:

$$(x-a)^{\alpha}(D^{\alpha}_{a|x})^2 f(x) - D^{\alpha}_{a|x} f(x) = \frac{x-a+(x-a)^{\alpha}}{1-x+a}. \quad (53)$$

The right hand side can be expanded as:

$$\frac{x-a+(x-a)^{\alpha}}{1-x+a} = \sum_{k=1}^{\infty}(x-a)^k + \sum_{k=0}^{\infty}(x-a)^k(x-a)^{\alpha}. \quad (54)$$

It is of index α and index $\beta = 1$. It is $1 \cdot \alpha$ fractionally-differentiable. So $f(x)$ is $2 \cdot \alpha$ fractionally-differentiable. We can write

$$\begin{aligned} f(x) &= c_{00} + c_{10}(x-a)^{\alpha} + c_{20}(x-a)^{2\alpha} \\ &+ 0 + c_{11}(x-a)^{\alpha+1} \\ &+ c_{02}(x-a)^2 \\ &+ \sum_{m\alpha+n > 3\alpha, (m,n)\neq(0,2)} c_{mn}(x-a)^{m\alpha+n}. \quad (55) \end{aligned}$$

Here we assume $0 < \alpha < 1$.

Or more manageably,

$$\begin{aligned} f(x) &= c_{00} + c_{10}(x-a)^{\alpha} \\ &+ \sum_{n=2}^{\infty} c_{0n}(x-a)^n + \sum_{n=1}^{\infty} c_{1n}(x-a)^{\alpha+n} \\ &+ \sum_{n=0}^{\infty} c_{2n}(x-a)^{2\alpha+n} + \sum_{m\geq 3}\sum_{n=0}^{\infty} c_{mn}(x-a)^{m\alpha+n}. \quad (56) \end{aligned}$$

Substituting (54) and (56) into (53), one gets

$$
\begin{aligned}
c_{mn} &= 0, \qquad m \geq 3; \\
1/c_{2n} &= \frac{\Gamma(n+2\alpha+1)}{\Gamma(n+1)} - \frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}; \\
1/c_{1n} &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n-\alpha+1)} - \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}, \quad n \geq 1; \\
c_{10} &= 0; \\
c_{0n} &= 0, \quad n \geq 1; \quad c_{00} = f(a).
\end{aligned} \tag{57}
$$

Thus we solved the fractional differential equation (53).

The above procedure can apply generally to fractional differential equations. For a generic fractional differential equation $F[(x-a), y(x-a), D^{\alpha}_{a|x}] = 0$, we can solve it by the above procedure summarized as follows.

1. Expand the $(x-a)$ part of the equation and find out the indices of $y(x-a)$,
2. Find out the number N such that $y(x-a)$ is $N \cdot \alpha$ fractionally-differentiable,
3. Expand $y(x-a)$,
4. Substitute the expansion series into the equation,
5. Find out the coefficients of the expansion series of $y(x-a)$.

We give another example in the following. Consider the fractional differential equation:

$$D^{\alpha}_{a|x} y(x) + f(x) y(x) = g(x), \tag{58}$$

where $f(x)$ and $g(x)$ are given functions of index β,

$$f(x) = f_0 + f_1 \cdot (x-a)^{\beta} + f_2 \cdot (x-a)^{2\beta} + f_3 \cdot (x-a)^{3\beta} + ..., \tag{59}$$

$$g(x) = g_0 + g_1 \cdot (x-a)^{\beta} + g_2 \cdot (x-a)^{2\beta} + g_3 \cdot (x-a)^{3\beta} + ..., \tag{60}$$

where f_i and g_i $(i = 0, 1, 2, 3, ...)$ are constants.

So $y(x)$ have two indices α and β. If $g_1 \neq 0$ and $\alpha < \beta < 2\alpha$, $g(x)$ is $1 \cdot \alpha$ fractionally-differentiable, then $y(x)$ is $2 \cdot \alpha$ fractionally-differentiable. We can write

$$\begin{aligned}
y(x) &= c_{00} + c_{10}(x-a)^{\alpha} + c_{20}(x-a)^{2\alpha} \\
&+ 0 + c_{11}(x-a)^{\alpha+\beta} \\
&+ c_{02}(x-a)^{2\beta} \\
&+ \sum_{m\alpha+n\beta>3\alpha,(m,n)\neq(0,2)} c_{mn}(x-a)^{m\alpha+n\beta}.
\end{aligned} \tag{61}$$

Or more manageably,

$$\begin{aligned}
y(x) &= c_{00} + c_{10}(x-a)^{\alpha} \\
&+ \sum_{n=2}^{\infty} c_{0n}(x-a)^{n\beta} + \sum_{n=1}^{\infty} c_{1n}(x-a)^{\alpha+n\beta} \\
&+ \sum_{n=0}^{\infty} c_{2n}(x-a)^{2\alpha+n\beta} + \sum_{m\geq 3}\sum_{n=0}^{\infty} c_{mn}(x-a)^{m\alpha+n\beta}.
\end{aligned} \tag{62}$$

Substituting (59), (60) and (62) into (58), one gets

$$\begin{aligned}
c_{00} &= y(a), \qquad\qquad c_{0n} = 0, \quad n \geq 1; \\
c_{1n} &= \frac{\Gamma(n\beta+1)}{\Gamma(\alpha+n\beta+1)}(g_n - f_n c_{00}); \\
c_{2n} &= -\frac{\Gamma(\alpha+n\beta+1)}{\Gamma(2\alpha+n\beta+1)}(f_n c_{10} + c_{1n}); \\
c_{mn} &= -\frac{\Gamma((m-1)\alpha+n\beta+1)}{\Gamma(m\alpha+n\beta+1)} c_{(m-1)n} \\
&= (-1)^{m-2}\frac{\Gamma(2\alpha+n\beta+1)}{\Gamma(m\alpha+n\beta+1)} c_{2n}, \qquad m \geq 3.
\end{aligned} \tag{63}$$

Thus we solved the fractional differential equation (58).

In this section we have discussed functions with two fractional indices, but the extension to functions with more fractional indices will not be difficult.

6 Summary

In summary, in this paper we introduced fractional series expansion method to fractional calculus. We defined a kind of fractional Taylor series of infinitely fractionally-differentiable functions. Based on our definition we generalized hypergeometric functions

and derived their differential equations. For finitely fractionally-differentiable functions, we observed that the non-infinitely fractionally-differentiability is due to more than one fractional indices. We expanded functions with two fractional indices and illustrated how this kind of series expansion can help to solve fractional differential equations.

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